International Journal of Theoretical Physics, Vol. 8, No. 6 (1973), pp. 409–417

Dirac Monopoles for General Gauge Theories

P. KLIMO and J. S. DOWKER

The Schuster Laboratory, University of Manchester, Manchester 13, *England*

Received: 21 *January* 1973

Abstract

This paper develops a non-local potential formalism for general gauge theories. With the help of this mathematical apparatus an argument for quantisation of the generalised charge is given, assuming that the Dirac monopoles are present.

Introduction

It is well known that in the Maxwell theory of the electromagnetic field the electromagnetic potentials can be expressed non-locally through the quantities of the electromagnetic field tensor to which they correspond.† The non-locality in this case is represented by the path dependence of the potentials, which is closely related to the invariance of the electromagnetic field under the gauge transformation of the local potentials in the following way: Any change of the path of the non-local potentials corresponds to a gauge transformation performed on the potentials in the local theory.

In the quantum mechanics, where the potentials of the electromagnetic field seem to play an essential role (e.g. Aharanov–Bohm effect), to some extent the non-local formulation allows a different view of the problem of the quantisation of the electromagnetic field to be taken (Mandelstam, 1962). The theory seems to have some advantage in attacking the problem of the quantisation of the magnetic monopole even if the physical side is somewhat overshadowed by the body of the mathematical apparatus.

In what follows we try to formulate a consistent non-local potential formalism of the non-linear generalisation of the Maxwell theory. The results can be applied directly to the theory of the Yang–Mills field and perhaps, with some modification, to the theory of gravitation as well. As an illustration a non-linear generalisation of the magnetic monopole is

† The non-local potentials for classical electrodynamics have been discussed, for example in the paper by Rohrlich (1966).

sketched and it is observed that such a theory would impose certain conditions on the charges which are very much in analogy with the Dirac quantisation of the charge of the magnetic monopole in ordinary electromagnetic theory.

1. *Path-dependent Potentials*

We adhere to the commonly used notation, considering the wave function $\psi(x)$ as a vector in an abstract n-dimensional space R^n parameterised by a point x in an ordinary three-dimensional space of events E.

The wave function $\psi(x)$ is to be considered a vector given by the equivalence classes under the gauge transformation group. The representation of this group $S(x)$ acts linearly in the vector space R^n.

To define the parallel displacement of $\psi(x)$ along a line element dx in E a field of $n \times n$ matrices $\Gamma_\mu(x)$ (linear connection) is given. This is a vector in E, but under the gauge transformation $S(x)$ undergoes the change

$$\Gamma_\mu'(x) = S\Gamma_\mu S^{-1} + (\partial_\mu S)^- S^{-1}(x) \tag{1.1}$$

The infinitesimal displacement of $\psi(X)$ along dx is then

$$\delta\psi(x) = -\Gamma_\mu(x)\, dx^\mu \psi(x) \tag{1.2}$$

The vector ψ has the transformation property in R^n:

$$\psi'(x) = S(x)\psi(x) \tag{1.3}$$

In similar fashion to the usual tensor calculus one defines a gauge-tensor field to be any field $A(x)$ of $n \times n$ matrices which transform as

$$A'(x) = S(x)A(x)S^{-1} \tag{1.4}$$

A gauge derivative of a wave function is defined as a gauge vector field

$$\nabla_\mu \psi(x) = (\partial_\mu - \Gamma_\mu)\psi \tag{1.5}$$

where ∂_μ is a covariant derivative.

Applied to a gauge tensor $A(x)$

$$\nabla_\mu A(x) = \partial_\mu A(x) - [\Gamma_\mu; A(x)] \tag{1.6}$$

so as to preserve the distributive action of the symbol.

Given a gauge group, there is a restriction on the parallel displacement in the sense that any displacement along a curve in E is to be a representation of a gauge group in R^n. This results in $\Gamma_\mu dx^\mu$ being a member of the Lie algebra of the gauge group. Of course the largest group which can be embodied in this formalism has to have the full linear representation in R^n and Γ is left to be, in this case, an arbitrary $n \times n$ matrix field.

The integral form of the parallel displacement can be expressed with the help of the product integral (ordered integral)

$$\delta\psi(x_1) = \exp\left(\int_{x_2}^{x_1} \Gamma_\mu \, dx^\mu\right)\psi(x) \tag{1.7}$$

with the integration along a given curve in E.

If ψ is now displaced parallel around an infinitesimal loop enclosing the area $dx^{[\mu}\delta x^{\nu]}$ then, to the first order, it is possible to take

$$\delta\psi = \Gamma_{\mu\nu}(x)\, dx^\mu \, \delta x^\nu \, \psi(x) \tag{1.8}$$

where $\Gamma_{\mu\nu} = \partial_{[\mu}\Gamma_{\nu]} - [\Gamma_\mu;\Gamma_\nu]$ is the curvature gauge tensor.

We shall call Γ_μ and $\tilde{\Gamma}_\mu$ to be gauge related if there exists a gauge transformation $U(x)$ such that

$$\tilde{\Gamma}_\mu(x) = U\Gamma_\mu(x)\, U^{-1} + \partial_\mu U . U^{-1} \tag{1.9}$$

Obviously, if Γ and $\tilde{\Gamma}$ are gauge related then

$$\tilde{\Gamma}_{ij} = U\Gamma_{ij}\, U^{-1} \tag{1.10}$$

Consider a set $z^\mu = z^\mu(x,\tau)$ of regular parametric curves in E, such that each point x is given only one curve starting at x and running to infinity:

$$z(x,\tau): \tau + < 0; -\infty) \qquad \begin{matrix} z(x;0) = x \\ z(x;-\infty) = -\infty \end{matrix} \tag{1.11}$$

Define Γ and Γ^* to be path-related if

$$\Gamma_\mu{}^*(x) = \int_{-\infty}^{0} d\tau U^{-1}\, \Gamma_{ab}(z)\, U(z) \frac{\partial z^a(x,\tau)}{\partial x^\mu} \cdot \frac{\partial z^b(x,\tau)}{\partial \tau} \tag{1.12}$$

where $U(z(x,\tau) = \exp(\int_{-\infty}^{z} \Gamma_\mu dx^\mu)$ is the displacement along the path $z(x,\tau)$.

We shall prove that if Γ^* and Γ are path-related they are also gauge-related and vice versa.

Proof. It is to be noticed at first that $\Gamma_\mu{}^*$ is a vector in E, however under the gauge transformation $S(x)$ with the boundary condition $\lim_{|x|\to\infty} S = 1$ remains unchanged. Due to the vector character of $\Gamma_\mu{}^*$ we are allowed to choose a special coordinate system (t_1, t_2, t_3) in E such that the set of the regular curves $z(x,\tau)$ serves as the set of the coordinate lines, say $t_2 = \text{const}$, $t_3 = \text{const}$, but otherwise arbitrary. In this system (dashed)

$$\acute{\Gamma}_1(t_1, t_2, t_3) = 0$$

$$\acute{\Gamma}_2(t_1, t_2, t_3) = \int_{-\infty}^{t_1} d\tau U^{-1}(\tau)\, \acute{\Gamma}_{21}(\tau, t_2, t_3)\, U(\tau)$$

$$\acute{\Gamma}_3(t_1, t_2, t_3) = \int_{-\infty}^{t_1} d\tau U^{-1}(\tau)\, \acute{\Gamma}_{31}(\tau, t_2, t_3)\, U(\tau)$$

where

$$U(\tau) = \exp\left\{ \int_{-\infty}^{\tau} \check{\Gamma}_1(s; t_2; t_3)\, ds \right\} \tag{1.13}$$

In succession to this we perform a gauge transformation in R^n to the new symbols $\check{K}$:

$$\begin{aligned} &\check{\Gamma}_1 \to \check{K}_1 = 0 \\ &\check{\Gamma}_2 \to \check{K}_2 \\ &\check{\Gamma}_3 \to \check{K}_3 \end{aligned} \tag{1.14}$$

It is already seen that the matrix of that transformation is $U(t)$ with the boundary condition $U(-\infty) = 1$.

Hence we are left with

$$\begin{aligned} &\check{\Gamma}_1{}^* = 0 \\ &\check{\Gamma}_2{}^* = \int_{-\infty}^{t_1} d\tau \check{K}_{21}(\tau; t_2; t_3) \\ &\check{\Gamma}_3{}^* = \int_{-\infty}^{t_1} d\tau \check{K}_{31}(\tau; t_2; t_3) \end{aligned} \tag{1.15}$$

which are identities if and only if

$$\begin{aligned} &\Gamma_1{}^* = K_1 \neq 0 \\ &\Gamma_2{}^* = K_2 \\ &\Gamma_3{}^* = K_3 \end{aligned} \tag{1.16}$$

But K and Γ are already gauge-related hence so are Γ and Γ^*.

It is seen that the path-related quantities are a natural generalisation of the non-local potentials in the Abelian-gauge theory.

One can rewrite the above theory in the formalism of the four-dimensional space-time of events E by merely increasing the number of indices.

2. *Monopoles and Strings*

The apparatus of the first section is employed to describe a physical situation. This is done by writing down the equations of motion for the curvature.

$$\begin{aligned} &\nabla_\mu \Gamma_{\mu\nu} = e_\nu \\ &\nabla_{[\mu} \Gamma_{\nu\rho]} = 0 \end{aligned} \qquad \mu, \nu = 0, 1, 2, 3 \tag{2.1}$$

where e_i, $e_0 \equiv e$ is electrical current and charge respectively. As mentioned before, the present discussion ignores the time dependence of the field but, if required, it could be trivially embodied into the formalism.

The first set of equations of motion represents a set of non-linear differential equations for Γ_μ so as to make them reflect the physical reality.

The second set is merely a condition stating that $\Gamma_{\mu\nu}$ is a curvature. Flirting, as we are, with the idea to set the right-hand side of the second set different from zero at the points of magnetic charge, we face immediately the task of restoring the consistency of the whole set. The problem is more complex than in an Abelian-gauge theory. The additional difficulty is how to define the gauge derivative.

One way how to approach this problem is to introduce Dirac's strings (Dirac, 1948).

We write the new equations in the form

$$\begin{aligned} \nabla_\mu R_{\mu\nu} &= e_\nu \\ \nabla_{[1} R_{23]} &= g \end{aligned} \tag{2.2}$$

The solution $R_{\mu\nu}$ is to be a gauge tensor, antisymmetric in indices μ and ν. This tensor is to be written as a sum of two gauge-tensor fields.

$$R_{\mu\nu} = \Gamma_{\mu\nu} + G_{\mu\nu} \tag{2.3}$$

$\Gamma_{\mu\nu}$ being in the form of a curvature

$$\Gamma_{\mu\nu} = \partial_{[\mu} \Gamma_{\nu]} - [\Gamma_\mu ; \Gamma_\nu] \tag{2.4}$$

and $G_{\mu\nu}$ a singular field which takes the non-vanishing values only along a line starting at the charge g and running into infinity (string). The covariant derivative is then defined as

$$\nabla_\mu = \partial_\mu - \Gamma_\mu \tag{2.5}$$

everywhere, including the line of the string. The set of the equation of motion is equivalent to

$$\begin{aligned} \nabla_\mu \Gamma_{\mu\nu} &= -\nabla_\mu G_{\mu\nu} + e_\nu \\ \nabla_{[1} G_{23]} &= g \end{aligned} \tag{2.6}$$

With the help of the path-related potentials we shall determine the singular gauge tensor field $G_{\mu\nu}$ to be

$$G_{\alpha\beta} = g\varepsilon_{\alpha\beta\gamma} \frac{dz^\gamma(x_0;\tau)}{\partial\tau} \qquad (\alpha, \beta, \gamma = 1, 2, 3) \tag{2.7}$$

where $\partial z^\alpha(x_0 \tau)/\partial\tau$ is the tangent at the point $z(x_0, \tau)$ to the string starting at the charge g located at the point X_0.

We proceed to define the quantity $\tilde{\Gamma}_\mu$

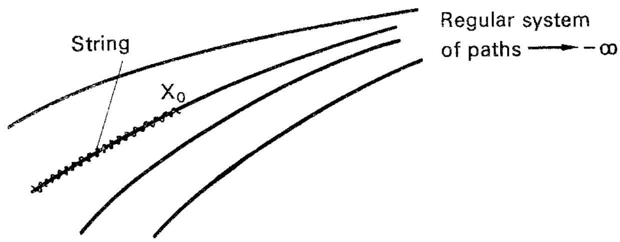
$$\tilde{\Gamma}_\mu(x) = \int_{-\infty}^{0} U^{-1} R_{ab}(z)\, U(z) \frac{\partial z^a(x,\tau)}{\partial x^\mu} \frac{\partial z^b(x,\tau)}{d\tau}\, \partial\tau \tag{2.8}$$

This is in a sense a generalisation of the path-related quantity $\Gamma_\mu{}^*$ defined in the first section. We assumed that $R_{\mu\nu}$ can be written in the form of the curvature $\Gamma_{\mu\nu}$ almost everywhere. Hence, taking into account the results of the previous section, Γ_μ and $\tilde{\Gamma}_\mu$ are gauge-related everywhere except on the string. By now we still have one degree of 'freedom' in hand: We did not specify the position of the string yet. This is simply done by defining Γ_μ and $\tilde{\Gamma}_\mu$ to be gauge-related everywhere, including the string.

As the next step we prove the identity:

$$\tilde{\Gamma}_{\mu\nu}(x) = U^{-1}(x)\,R_{\mu\nu}(x)\,U + \varepsilon_{\mu\nu\alpha}\frac{dz^\alpha(x_0\,\tau)}{\partial\tau}\,U^{-1}g(x_0)\,U \tag{2.9}$$

where $U(x)$ is the parallel displacement along the path leading to x, $\varepsilon_{\mu\nu\alpha}$ the completely antisymmetric tensor and x_0 the location point of the magnetic charge g. It is to be noticed that $\Gamma_{\mu\nu}(x)$ is singular along the path from x_0 which is to be identified with the string.



To prove the above identity we transform (2.8) and (2.7) to a new coordinate system (t_1, t_2, t_3) in E such that the set of the paths becomes the set of the coordinate lines $t_2 = \text{const}$, $t_3 = \text{const}$. Afterwards we perform a gauge transformation with $U(t_1)$ and in agreement with the discussion in the first section we obtain the transformed equations (2.8) and (2.9) in the form

$$\begin{aligned}
\tilde{\Gamma}_1(t_1;t_2;t_3) &= 0 \\
\tilde{\Gamma}_2(t_1;t_2;t_3) &= \int_{-\infty}^{t_1} R_{21}(\tau;t_2;t_3)\,d\tau \\
\tilde{\Gamma}_3(t_1;t_2;t_3) &= \int_{-\infty}^{t_1} R_{31}(\tau;t_2;t_3)\,d\tau
\end{aligned} \tag{2.10}$$

$$\begin{aligned}
\tilde{\Gamma}_{21} &= R_{21} \\
\tilde{\Gamma}_{31} &= R_{31} \\
\tilde{\Gamma}_{32} &= R_{32} + g
\end{aligned} \tag{2.11}$$

The first two equations of the set (2.11) are easily seen to be identities. To show that the same is true for the remaining one we calculate

$$\begin{aligned}
&\partial_3\Gamma_2(t;t_2;t_3) - \partial_2\Gamma_3(t;t_2;t_3) \\
&\quad = \int_{-\infty}^{t} (\nabla_3 R_{21} - \nabla_2 R_{31})\,d\tau + \int_{-\infty}^{t} ([\Gamma_3;R_{21}] - [\Gamma_2;R_{31}])\,d\tau
\end{aligned} \tag{2.12}$$

where the arguments of the integrands are taken at the point $(\tau; t_2; t_3)$. But according to our definition the original Γ and $\tilde{\Gamma}$ in (2.8) are gauge-related. This amounts to $\tilde{\Gamma} = \Gamma$ in the new gauge. With the help of (2.8) the first bracket of the second integral in (2.12) can be evaluated by parts and the second integral is seen to be equal to $[\tilde{\Gamma}_2; \tilde{\Gamma}_3]_{(t_1; t_2; t_3)}$. Hence

$$\tilde{\Gamma}_{23} = \int_{-\infty}^{t_1} d\tau (\nabla_\tau R_{23}(\tau; t_2; t_3) + g\,\delta(\tau - t_0)) \tag{2.13}$$

where we used the equation of motion

$$\nabla_{[1} R_{23]} = g \tag{2.14}$$

But $\nabla_\tau = \partial_\tau - \Gamma_1 = \partial_\tau$ in the new gauge and the last equation of (2.11) is seen ready to be an identity. Going back to (2.9) and using the fact that $\tilde{\Gamma}_\mu$ and Γ_μ are gauge-related, the singular field $G_{\mu\nu}$ is determined by (2.7). This completes our proof and indicates that the above-given theory with strings is self-consistent.

3. *Quantum Condition for Generalised Charge*

Before proceeding to give an argument for quantisation of charges in the theory presented, we wish to add one or two comments in addition to the above discussion on path-dependent potentials.

The path-related quantities $\Gamma_\mu{}^*$ have been defined on the paths extending to infinity. Obviously there is no need to dwell upon that view; indeed, any set of regular paths would do. Especially if we consider a closed curve in a regular plane, then the parallel displacement of the wave function $\psi(x)$ along the curve, starting at x_0, is given by

$$\delta\psi = \exp\{\oint \Gamma_\mu dx^\mu\}\,\psi = \exp\left\{\iint U^{-1} \Gamma_{\mu\nu} U\, dx^\mu dx^\nu\right\} \tag{3.1}$$

where U is the displacement along the set of regular paths streaming out of point x_0; the first of the two integrals being an ordered one. This expression is essentially the same as the one derived by Schlesinger (1927).

We give the path-dependent theory its final shape: Charges are described by the path-dependent wave function $\psi(x,L)$ (the path L is independent of the set of parametric paths used in the definition of the path-like potentials). According to Mandelstam (1962) we define a path derivative, being a change of $\psi(x,L)$ along the path L:

$$\delta_\nu \psi(x,L) = \lim_{dx \to 0} \frac{\psi(x, L + dx) - \psi(x,L)}{dx^\nu} \tag{3.2}$$

the equation of motion for charges (Schrödinger equation) is

$$i\,\partial_t \psi(x,L) = -\delta_\nu \delta_\nu \psi(x,L) \tag{3.3}$$

The link between a path-dependent formalism and the local one may be established by defining

$$\psi(x,L) = \exp\left\{ie \int_L \Gamma_\mu \, dx^\mu\right\} \psi(x) \tag{3.4}$$

where e is the electric charge and $\psi(x)$ the local wave function. The field Γ_μ is to be determined from the field equations given in Section 2.

Let us now consider the situation when there is an electric charge moving in the field of a static magnetic monopole, located at the origin of the coordinate system. $\psi(x,L)$ is to be the wave function of the electric charge alone and the field Γ_μ will be determined from the spherically symmetric solution $R_{\mu\nu}$ of the field equations.

$$\begin{aligned} \nabla_\mu R_{\tilde{\mu}\nu} &= 0 \\ \nabla_{[1} R_{23]} &= g \end{aligned} \tag{3.5}$$

Let us try to find such a solution on account of symmetry alone.

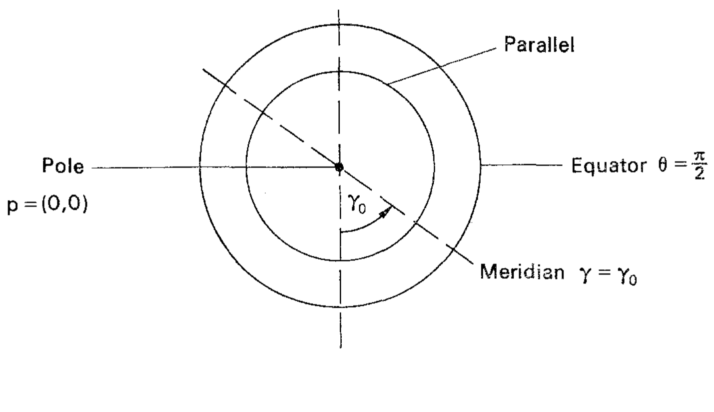
For this purpose we introduce the polar coordinates (r, θ, ρ). From the indistinguishability of the different directions along the sphere $r = \text{const}$, $t = \text{const}$, it follows that

$$R_{\theta r} = R_{\theta t} = R_{\rho t} = 0 \tag{3.6}$$

Now choose the point $\theta = \rho = 0$ on this sphere to be a pole, the set of meridians through it as the set of regular paths with help of which we can define the path-related potentials. Of course, each path is a geodesic and at any point (r, θ, ρ) the field $R_{\theta\rho}$ is equivalent to the field at $(r, 0, 0)$ in the sense that it is obtained from one point to another by a parallel displacement along the meridian

$$R_{\theta\rho}(\theta; \rho) = U(\theta; \rho)\, R_{\theta\rho}(0; 0)\, U^{-1}(\theta; \rho) \tag{3.7}$$

With the help of the integral formula (3.1) we are in the position to evaluate the parallel displacement of ψ along the equator of the sphere.



This is given by

$$\delta\psi = \exp\left(i\mathrm{e}R_{\theta\rho}(0,0)\,2\pi r^2\right)\psi \tag{3.8}$$

where the ordered integral along the equator disappeared as the field $R_{\theta\rho}$ along any parallel is constant. We now take advantage of an argument due to Loos (1965) by which the parallel displacement of a wave function along a great circle is identified. This is because of the spherical symmetry of the field.

Hence the restriction

$$\mathrm{e}R_{\theta\rho}(0,0) = Kr^{-2} \tag{3.9}$$

where K is the solution of the equation

$$1 = \exp(i2\pi K) \tag{3.10}$$

To find the physical interpretation of K we calculate $e\nabla_r R_{\theta\rho}$. This is done most easily in a gauge where $\Gamma_r' = 0$. In this new gauge

$$\acute{\mathrm{e}}\grave{\nabla}_r \grave{R}_{\theta\rho}(0,0) = \grave{K}\,\delta(r)$$

where $\grave{K} = UKU^{-1}$; $\dot{\mathrm{e}} = U\mathrm{e}U^{-1}$ and U is the same matrix as before. Looking back to the equation of motion

$$\grave{\nabla}_{[1}\grave{R}_{23]} = \dot{\mathbf{g}}$$

$\grave{K}$ is to be identified with the product $\dot{g}\,.\,\dot{e}$ of the magnetic and electric charges. The quantisation condition for $\mathbf{g}$ and e then reads

$$1 = \exp(i2\pi\mathbf{g}\,.\,\mathrm{e}) \tag{3.11}$$

Conclusion

The argument for the quantisation of charge in this work is not the same as the one given by Dirac (1948). In Dirac's theory a string must never pass through a charged particle. This is a necessary condition if we require the equation of motion to follow from an action principle. The space thus becomes multiply connected. Postulating that the wave function be single-valued, Dirac derives the quantum condition for charge.

In our work the strings can pass through any region of the space. The quantisation follows from the spherical symmetry of the field produced by a singular monopole. Strings are employed for only one purpose, to write down the equation of the field in the presence of monopoles. Whether both arguments are two sides of the same coin is not clear.

References

Dirac, P. A. M. (1948). *Physical Review*, **74**, 817.
Loos, H. G. (1965). *Nuclear Physics*, **72**, 677.
Mandelstam, S. (1962). *Annals of Physics*, **19**, 1.
Rohrlich, F. (1966). *Physical Review*, **150**, 1104.
Schlesinger, L. (1927). *Mathematische Annalen*, **99**, 413.